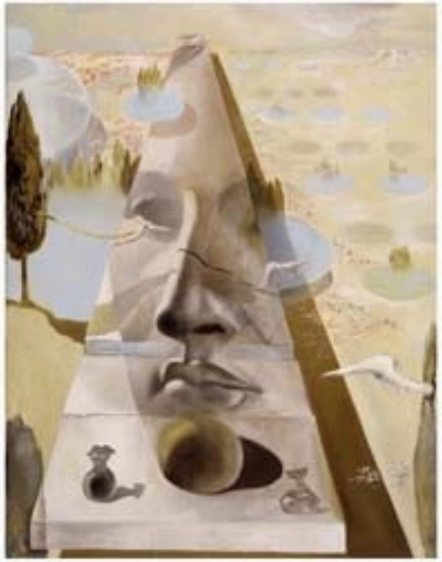
CS231A

Computer Vision:
From 3D Reconstruction to Recognition



Optimal Estimation

Perception as a Continuous Process

Perception as a Multi-Modal Experience

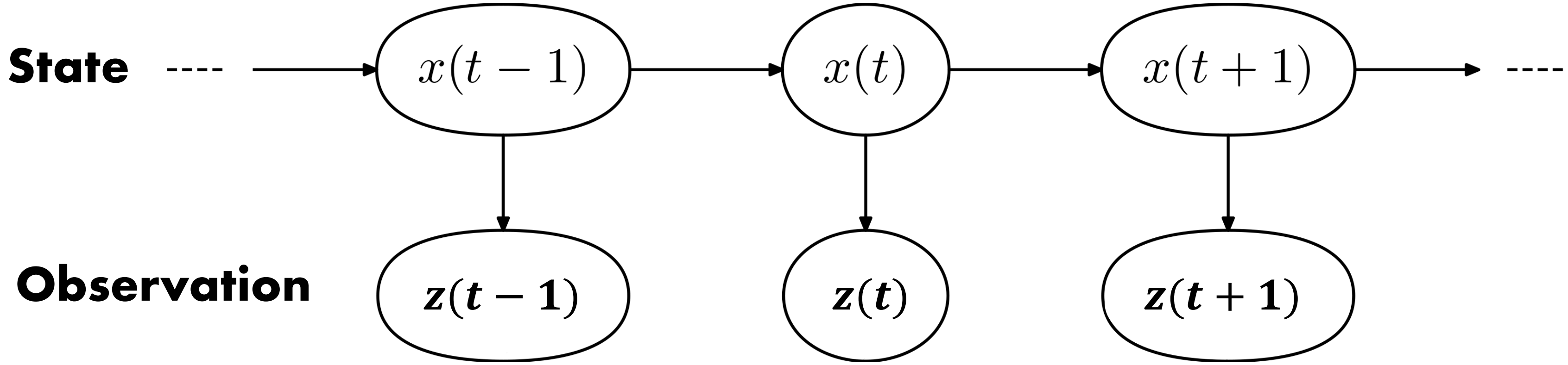
Perception as Inference

Recursive State Estimation

Mathematical Formalism to:

- continuously integrate measurements
- from different sensor sources
- to infer the state of a latent variable

What is a state? What is a representation?



Hidden Markov Model

Representations for Autonomous Driving

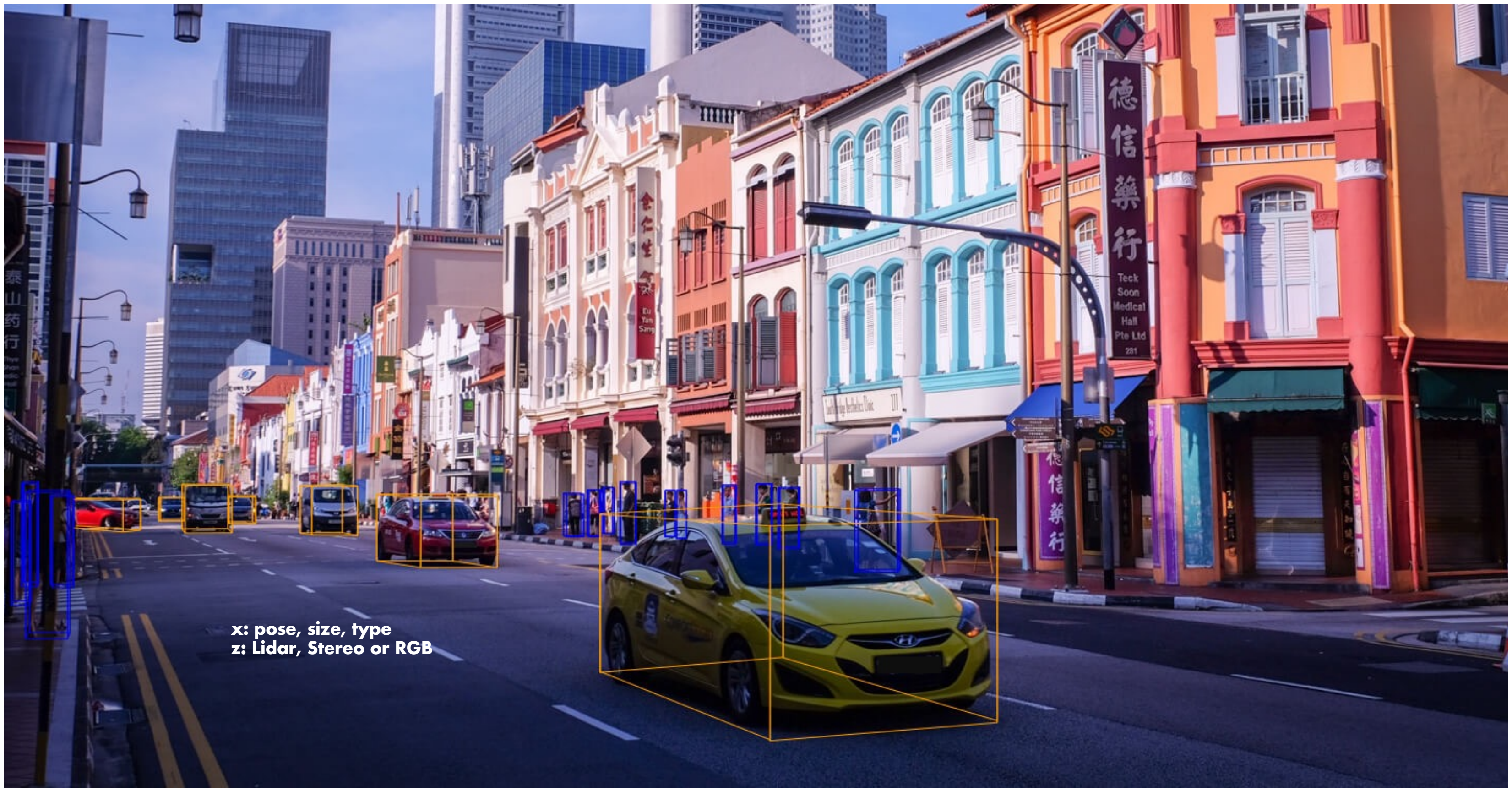
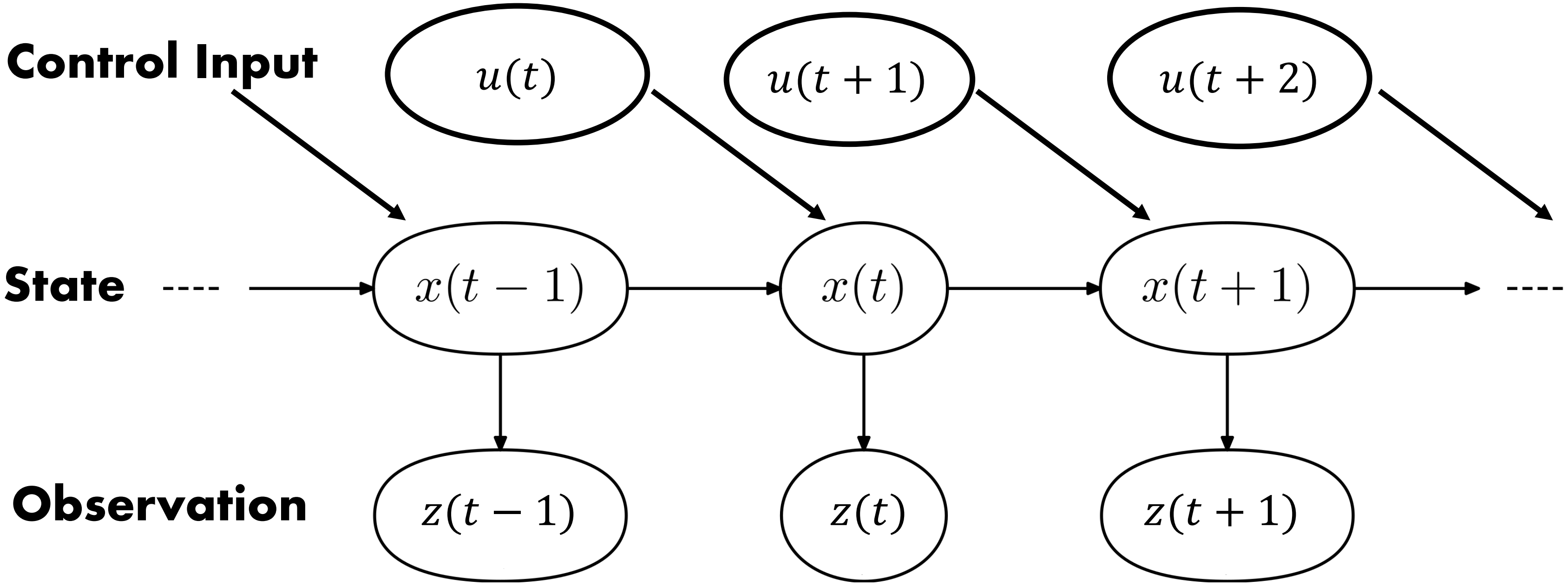


Image adapted from NuScenes by Motional. nuscenes.org

Representations for Manipulation

Manuel Wühtrich et al. "Probabilistic Object Tracking using a Depth Camera", IROS 2013

Why do we care about state estimation in Robotics?



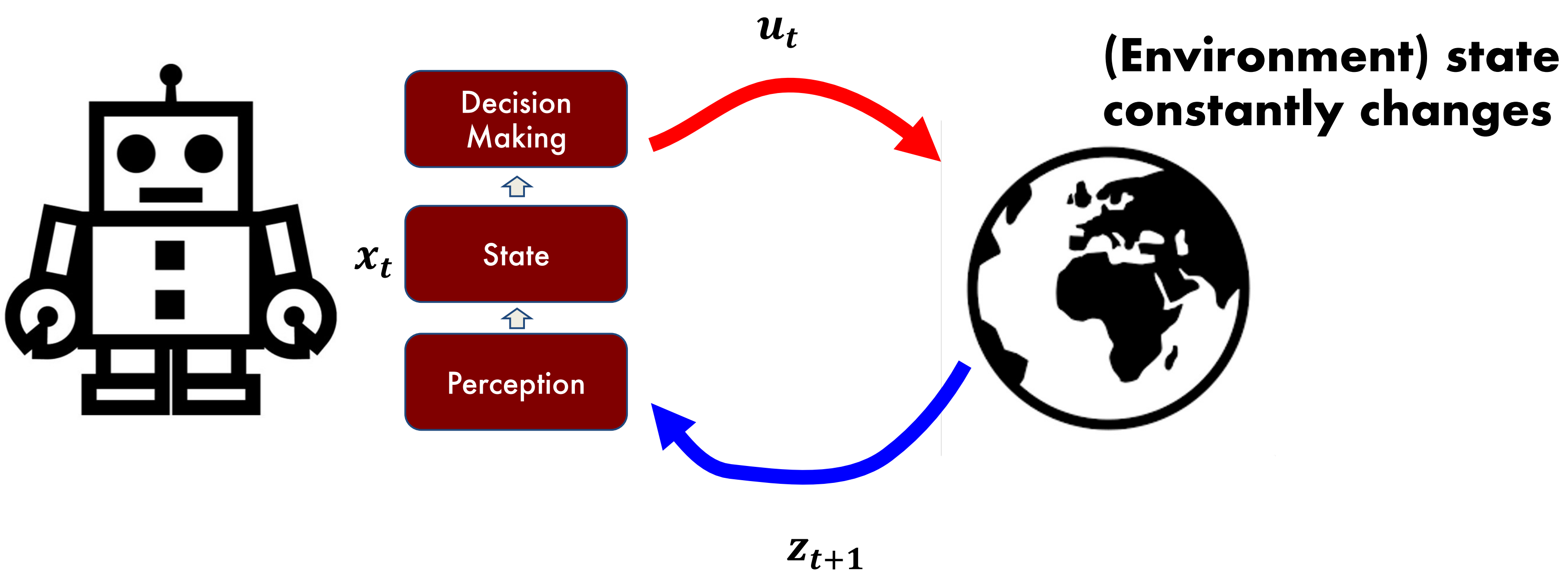
Partially Observable Markov Decision Process

Today

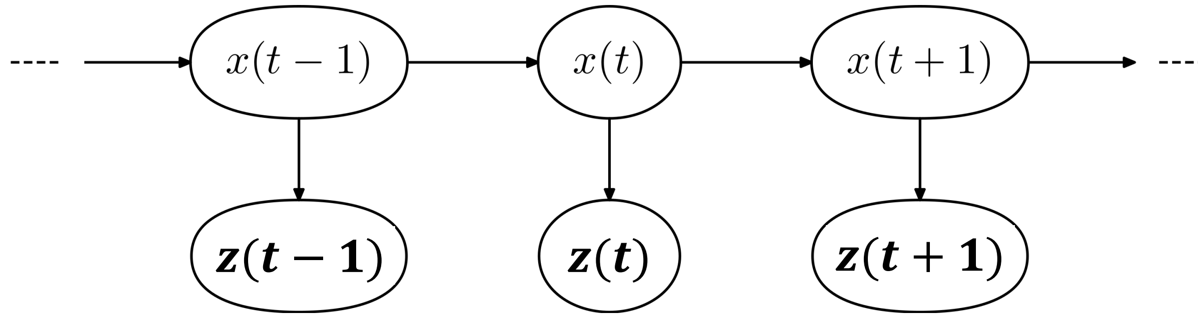
- Intro: Why state estimation?
- Bayes Filter
- Kalman Filter
- Extended Kalman Filter

- For more depth:
  - **AA 273: State Estimation and Filtering for Robotic Perception – Mac Schwager**

The Agent and the Environment



Notation



x	State of dynamical system, dim n
x_t	Instantiation of system state at time t
z	Sensor Observation Vector, dim k
z_t	Specific Observation at time t
u	Robot action / control input, dim m
u_t	Robot action / control input at time t
$p(x_t \mid z_{0:t}, u_{0:t})$	Probability distribution

Markov Assumption

State is complete

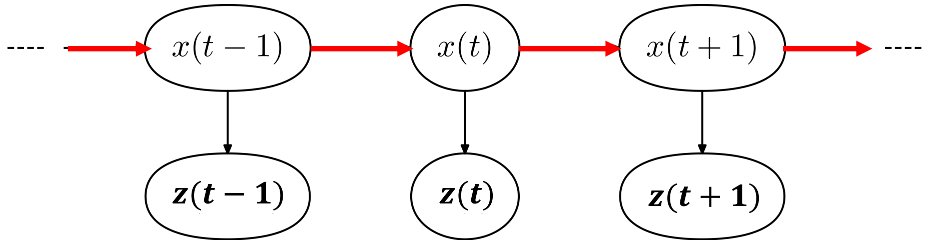
Probabilistic Generative Laws

- Evolution of state and measurement governed by probabilistic laws
- x_t generated stochastically

State Transition Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$$



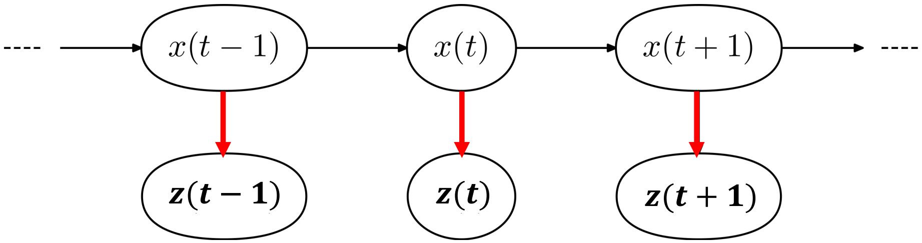
- Assumption: State complete

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

Measurement Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t})$$



- Assumption: State complete

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

Belief Distribution

- Assigns probability to each possible hypothesis about what the true state may be
- Posterior distributions over state conditioned on all the data

$$bel(x_t) \quad = \quad p(x_t \mid z_{1:t}, u_{1:t})$$

- Before incorporating measurement $\boldsymbol{z_t}$ = prediction

$$\overline{bel}(x_t) \quad = \quad p(x_t \mid z_{1:t-1}, u_{1:t})$$

The Bayes Filter

- Recursive filter for estimating x_t only from x_{t-1}, z_t and u_t and not from the ever-growing history $z_{1:t}, u_{1:t}$

Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:** Transition/Dynamics model

for all x_t do

$\overline{bel}(x_t) = \int \boxed{p(x_t \mid u_t, x_{t-1})}\, bel(x_{t-1})\, dx$ Predict Step

$bel(x_t) = \eta \boxed{p(z_t \mid x_t)}\, \overline{bel}(x_t)$ Update Step

endfor

return $bel(x_t)$

Measurement Model

Simple example – Belief & Measurement Model

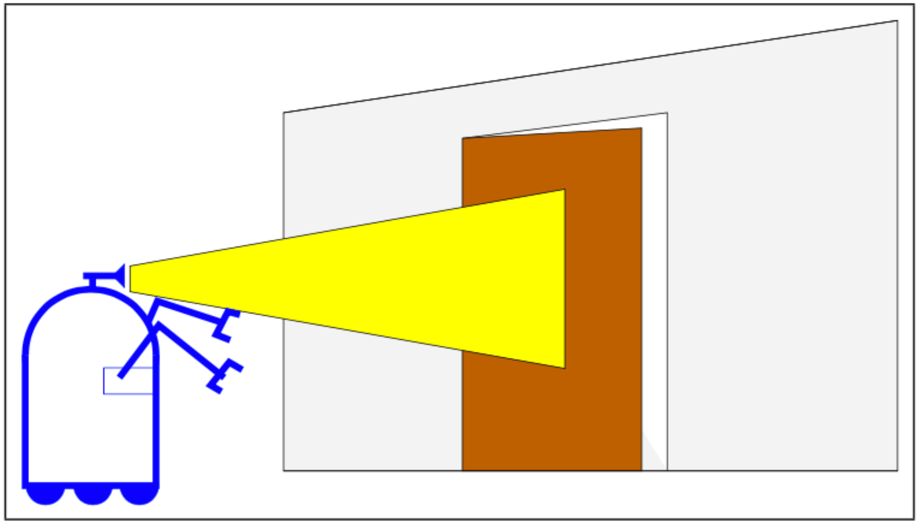


Figure 2.2 A mobile robot estimating the state of a door.

$$bel(X_0 = \textbf{open}) = 0.5$$
$$bel(X_0 = \textbf{closed}) = 0.5$$

$$p(Z_t = \textbf{sense\_open} \mid X_t = \textbf{is\_open}) = 0.6$$
$$p(Z_t = \textbf{sense\_closed} \mid X_t = \textbf{is\_open}) = 0.4$$

$$p(Z_t = \textbf{sense\_open} \mid X_t = \textbf{is\_closed}) = 0.2$$
$$p(Z_t = \textbf{sense\_closed} \mid X_t = \textbf{is\_closed}) = 0.8$$

Simple example – Transition Model

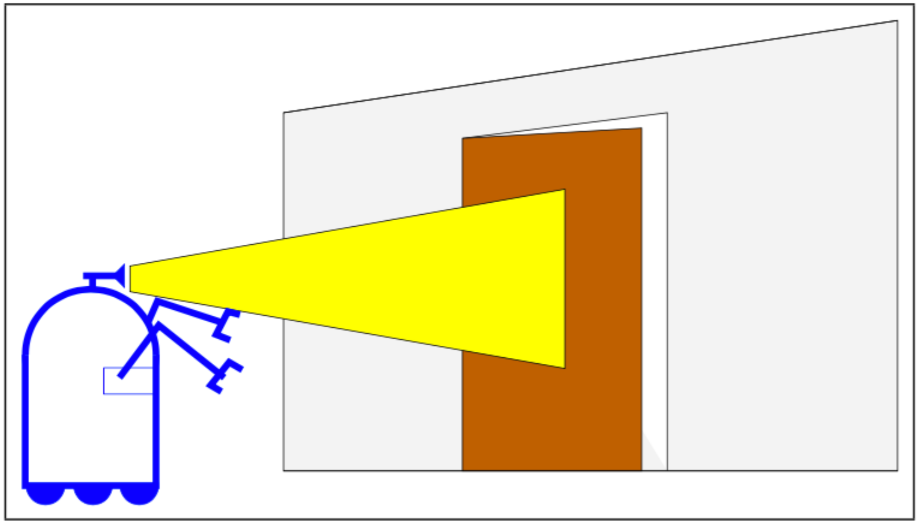


Figure 2.2 A mobile robot estimating the state of a door.

$$
\begin{aligned}
p(X_t = \textbf{is\_open} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_open}) &= 1 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_open}) &= 0 \\
p(X_t = \textbf{is\_open} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_closed}) &= 0.8 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_closed}) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(X_t = \textbf{is\_open} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_open}) &= 1 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_open}) &= 0 \\
p(X_t = \textbf{is\_open} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_closed}) &= 0 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_closed}) &= 1
\end{aligned}
$$

The Bayes Filter - Derivation

- Bayes Rule

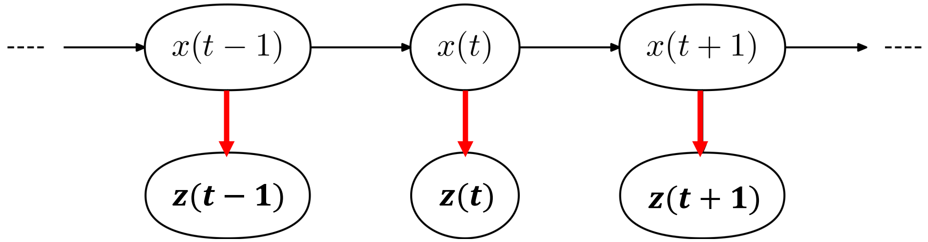
$$\boldsymbol{p(a,b) = p(a|b)p(b) = p(b|a)p(a)}$$

$$\boldsymbol{p(a|b) = \frac{p(b|a)p(a)}{p(b)}}$$

$$p(x_t \mid z_{1:t}, u_{1:t}) \quad = \quad \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\ p(x_t \mid z_{1:t-1}, u_{1:t})}{\boxed{p(z_t \mid z_{1:t-1}, u_{1:t})}}$$

Normalization

The Bayes Filter - Derivation



- State is complete

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

- Simplify

$$\begin{aligned} p(x_t \mid z_{1:t}, u_{1:t}) &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \\ &= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta\, \boxed{p(z_t \mid x_t)}\, p(x_t \mid z_{1:t-1}, u_{1:t}) \end{aligned}$$

simplified

The Bayes Filter - Derivation

$$p(x_t \mid z_{1:t}, u_{1:t}) = \eta\, p(z_t \mid x_t)\, \boxed{p(x_t \mid z_{1:t-1}, u_{1:t})}$$

$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

This still depends on entire history

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
  for all x_t do
    $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$
    $bel(x_t) = \eta\, \boxed{p(z_t \mid x_t)}\, \overline{bel}(x_t)$ **Update Step**
  endfor
  return $bel(x_t)$

Measurement Model

The Bayes Filter - Derivation

$$bel(x_t) \quad = \quad \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$$

- Total probability $\boldsymbol{p(a) = \int p(a|b)p(b)db}$

$$\begin{aligned} \overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \int \underline{p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})} \, \boxed{p(x_{t-1} \mid z_{1:t-1}, u_{1:t})} \, dx_{t-1} \end{aligned}$$

Previous Belief over x

- State is complete

$$\underline{p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})} \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

The Bayes Filter - Derivation

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

$$\begin{aligned}
\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \int \underbrace{\boxed{p(x_t \mid x_{t-1}, u_t)}}_{\text{simplified}}\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}
\end{aligned}$$

Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:**
for all x_t do
$\overline{bel}(x_t) = \int \boxed{p(x_t \mid u_t, x_{t-1})}\, bel(x_{t-1})\, dx$
$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
endfor
return $bel(x_t)$

Transition/Dynamics model

Predict Step

Limitations

1. $p(x)$ is defined $\forall x$ – intractable
   - Discrete and small spaces
   - Continuous and/or large spaces – Moments, Finite # of samples
2. The integral term -> costly to compute

The Bayes Filter

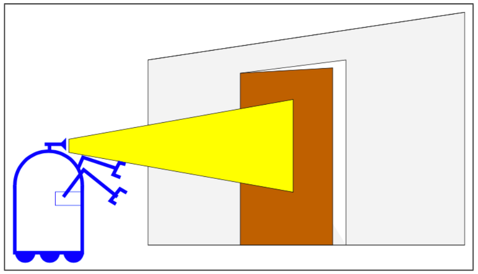


Figure 2.2 A mobile robot estimating the state of a door.

- Recursive filter for estimating x_t only from x_{t-1}, z_t and u_t and not from the ever-growing history $z_{1:t}, u_{1:t}$

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
  for all x_t do
    $\overline{bel}(x_t) = \int \boxed{p(x_t \mid u_t, x_{t-1})}\, bel(x_{t-1})\, dx$ **Predict Step**
    $bel(x_t) = \eta\, \boxed{p(z_t \mid x_t)}\, \overline{bel}(x_t)$ **Update Step**
  endfor
  return $bel(x_t)$

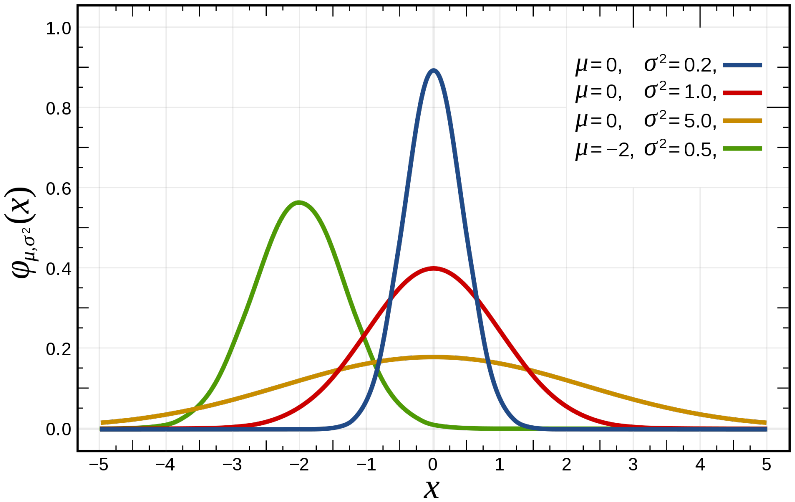
Transition/Dynamics model

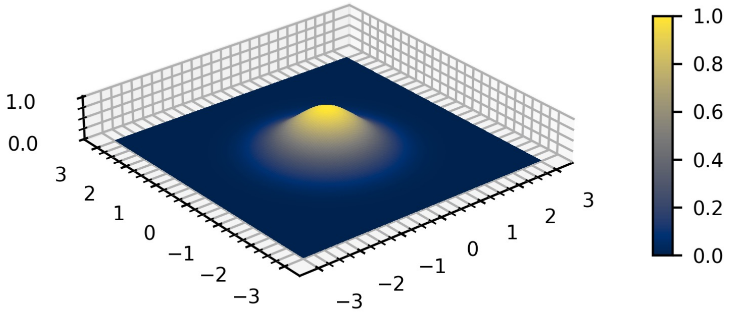
Measurement Model

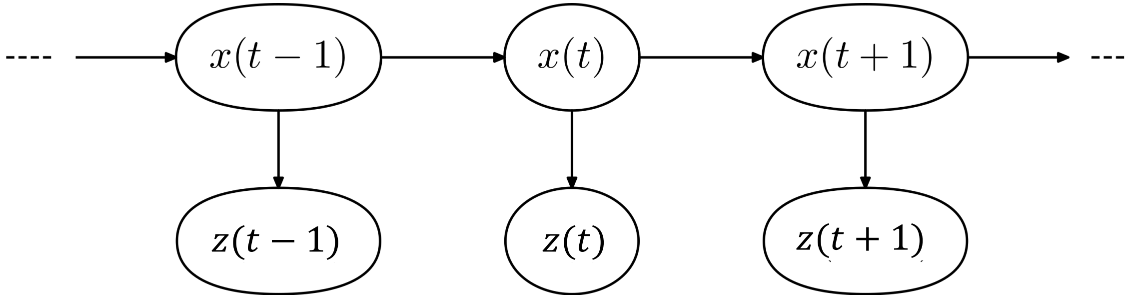
Gaussian Filters - Kalman Filter

$$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$p(x) \quad = \quad \det\left(2\pi\Sigma\right)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$







Kalman Filter

- Gaussian Belief
- Linear Transition Model

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

Process Noise $\varepsilon \sim N(0, R)$

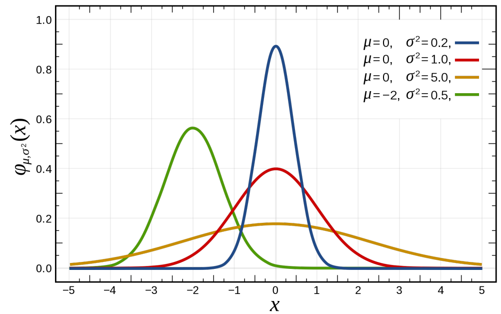
$$x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} \qquad u_t = \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{pmatrix}$$

- Linear Measurement Model

$$z_t = C_t x_t + \delta_t$$

Measurement Noise $\delta \sim N(0, Q)$

Kalman Filter



- Initial Belief $\mathbf{x_0} \sim N(\boldsymbol{\mu_0}, \boldsymbol{\Sigma_0})$

$$bel(x_0) = p(x_0) = \det(2\pi\Sigma_0)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_0-\mu_0)^T\Sigma_0^{-1}(x_0-\mu_0)\right\}$$

- Distribution over next state

$$p(x_t \mid u_t, x_{t-1})$$

$$= \det(2\pi R_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_t - \underline{A_t x_{t-1} - B_t u_t})^T R_t^{-1}(x_t - \underline{A_t x_{t-1} - B_t u_t})\right\}$$

Process Noise

Transition Model

- Likelihood of Measurement

Measurement Noise

$$p(z_t \mid x_t) = \det(2\pi Q_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(z_t - \underline{C_t\, x_t})^T Q_t^{-1}(z_t - \underline{C_t\, x_t})\right\}$$

Measurement Model

The Kalman Filter Algorithm

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)**:**

$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$

$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$

$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$

$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Uncertainty increases

K = Kalman Gain $K \approx \frac{R}{Q}$

Uncertainty decreases

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)**:**

for all x_t do

$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \, bel(x_{t-1}) \, dx$ **Predict Step**

$bel(x_t) = \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$ **Update Step**

endfor

return $bel(x_t)$

If R large, then K is large. Update dominated by innovation.

If Q large, then K is small. Update dominated by prediction.

Example

$p(z_0|x_0)$
Measurement

$p(x_0)$

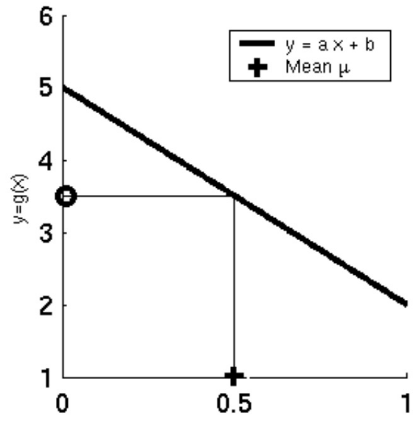
$bel(x_0)$
After Update

$\overline{bel}\,(x_1)$
After Prediction

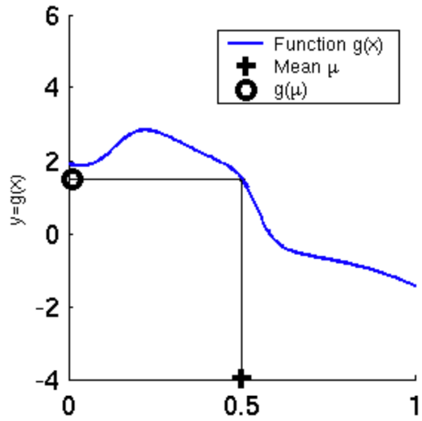
$p(z_1|x_1)$
Measurement

$bel(x_1)$
After Update

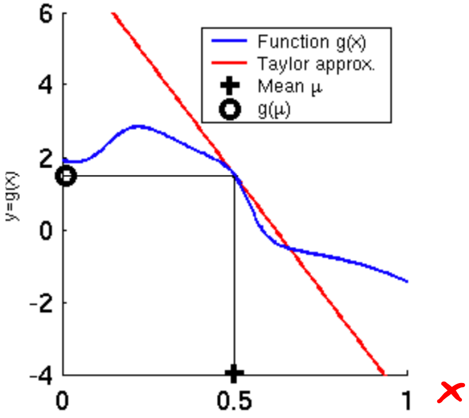
Propagating a Gaussian through a Linear Model



Propagating a Gaussian through a Non-Linear Model



Linearizing the Non-Linear Model



Representations for Manipulation

Manuel Wühtrich et al. "Probabilistic Object Tracking using a Depth Camera", IROS 2013

Extended Kalman filter - Process Model

$$
\begin{aligned}
x_t &= g(u_t, x_{t-1}) + \varepsilon_t \\
z_t &= h(x_t) + \delta_t \ .
\end{aligned}
$$

Process Model

Measurement Model

First order Taylor Expansion – linear approximation around value and slope

$$g'(u_t, x_{t-1}) \quad := \quad \frac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}}$$

Gradient of Nonlinear function around x_{t-1}

Taylor Expansion

$$
\begin{aligned}
g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=:\, G_t} (x_{t-1} - \mu_{t-1}) \\
&= g(u_t, \mu_{t-1}) + \underline{G_t \ (x_{t-1} - \mu_{t-1})}
\end{aligned}
$$

Jacobian

Extended Kalman filter - Process Model

$$\begin{aligned} g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=:\, G_t} (x_{t-1} - \mu_{t-1}) \\ &= g(u_t, \mu_{t-1}) + G_t\, (x_{t-1} - \mu_{t-1}) \end{aligned}$$

Same equations as in previous slide

Written as Gaussian:

$$\begin{aligned} & p(x_t \mid u_t, x_{t-1}) \\ & \approx \det(2\pi R_t)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\left[x_t - g(u_t, \mu_{t-1}) - G_t\, (x_{t-1} - \mu_{t-1})\right]^T \right. \\ & \quad \left. R_t^{-1} \left[x_t - g(u_t, \mu_{t-1}) - G_t\, (x_{t-1} - \mu_{t-1})\right]\right\} \end{aligned}$$

Extended Kalman Filter – Measurement Model

$$x_t = g(u_t, x_{t-1}) + \varepsilon_t$$ **Process Model**

$$z_t = h(x_t) + \delta_t \ .$$ **Measurement Model**

First order Taylor Expansion – linear approximation around value and slope

$$h(x_t) \approx h(\bar{\mu}_t) + \underbrace{h'(\bar{\mu}_t)}_{=:\, H_t} (x_t - \bar{\mu}_t)$$

$$= h(\bar{\mu}_t) + \underline{H_t \, (x_t - \bar{\mu}_t)}$$

Jacobian

Written as Gaussian:

$$p(z_t \mid x_t) = \det(2\pi Q_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \left[z_t - h(\bar{\mu}_t) - H_t \, (x_t - \bar{\mu}_t)\right]^T Q_t^{-1} \left[z_t - h(\bar{\mu}_t) - H_t \, (x_t - \bar{\mu}_t)\right]\right\}$$

The Extended Kalman Filter Algorithm

1: **Algorithm Extended_Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)**:**
2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$
3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$
4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$
5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$
7: return μ_t, Σ_t

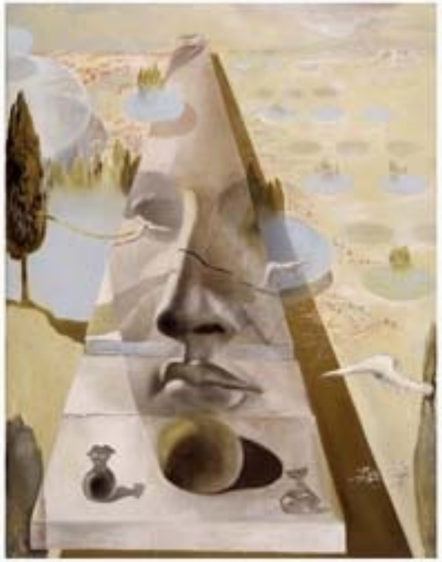
Predict

Update

	Kalman filter	EKF
state prediction (Line 2)	$A_t \, \mu_{t-1} + B_t \, u_t$	$g(u_t, \mu_{t-1})$
measurement prediction (Line 5)	$C_t \, \bar{\mu}_t$	$h(\bar{\mu}_t)$

CS231

Introduction to Computer Vision



Next lecture:

Optimal Estimation cont'